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AN APPROACH TO THE THEORETICAL FOUNDATIONS OF ELECTRONIC CIRCUIT DESIGN: SHEFFER STROKE HILBERT ALGEBRAS AND FUZZY SETS WITH THRESHOLDS

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Abstract. The Sheffer Stroke (also known as the NAND operator) is an important logical operator used in digital circuits based on the theory of fuzzy sets. The concept of fuzzy subalgebras (or fuzzy ideals) with thresholds in Sheffer Stroke Hilbert algebras is introduced. Several properties of these fuzzy subalgebras (fuzzy ideals) are discussed, and their generalizations are proven. Furthermore, the relationships between fuzzy subalgebras (fuzzy ideals) with thresholds and their level subsets are explored in detail.

Keywords: Sheffer stroke Hilbert algebra, fuzzy subalgebra with thresholds, fuzzy ideal with thresholds.
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1 Introduction

In the early 1950s, L. Henkin and A. Diego introduced Hilbert algebras, which serve as the algebraic counterpart to Hilbert's positive implicative propositional calculus, as discussed in Rasiowa (1974). These algebras were developed for the study of intuitionistic and other non-classical logics (Diego, 1966). Building on this foundation, further work by Busneag (1985, 1988, 1993) led to the definition of a bounded Hilbert algebra by Idziak. This algebra is a specific type of BCK-algebra that includes lattice operations, as outlined in Idziak (1984).

Electronic circuits are basic structures with a wide range of applications, from consumer electronics to industrial systems. Consisting of various electronic components such as resistors, capacitors, transistors and integrated circuits, these circuits play an important role in signal processing, power management and communication systems. In consumer electronics, it enables the use of functions such as data processing, display and wireless communication in devices such as smartphones, computers and televisions. Electronic circuits are indispensable for automation, control systems and robotics in industrial applications and increase efficiency and precision in production processes. Additionally, advances in electronic circuit design, including the development of low-power, high-speed circuits, continue to spur innovations in fields such as medical devices, renewable energy, and automotive systems, making electronic circuits indispensable in

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modern technology. As their use increases with the development of technology, studies on the theoretical foundations of such critically important structures are very important.

The Sheffer operation, also referred to as the Sheffer stroke or NAND operator, was introduced by Sheffer (1913). This operation is particularly significant because it is functionally complete on its own, meaning it can be used independently, without relying on any other logical operators, to build a logical system. In fact, any axiom of a logical system can be reformulated using only the Sheffer operation. This property makes it easier to analyze and manipulate the properties of the logical system constructed with it. Moreover, it's important to note that the axioms of Boolean algebra— which serves as the algebraic foundation for classical propositional logic— can be fully expressed using just the Sheffer operation. This underscores the fundamental and versatile role the Sheffer operation plays in both logical and algebraic structures.

By applying the Sheffer stroke operation to various logical algebras, it serves to reduce the axiom systems of these structures, leading to a variety of useful results in algebra, logic, and related fields. As a result, mathematicians have extensively studied algebraic structures involving the Sheffer stroke operation. These include Sheffer stroke basic algebras (Senturk & Oner, 2017), Sheffer stroke MTL-algebras (Senturk, 2020), Sheffer stroke BL-algebras (Oner et al., 2022), Sheffer stroke BCK-algebras (Oner et al., 2022), and Sheffer stroke UP-algebras (Oner et al., 2021a). There have been many papers published in recent years on these Sheffer stroke algebras, for example, Senturk et al. (2020); Senturk & Oner (2021); Senturk (2021, 2022); Senturk & Oner (2022); Kim et al. (2024); Bae Jun et al. (2024) and Chunsee et al. (2024).

The Sheffer stroke is an important logical operator used in digital circuits and engineering applications. In mathematical logic, the Sheffer Stroke is a binary operator that can be used to express all other basic logical operations, such as AND, OR, and NOT. This property makes it particularly useful for simplifying and optimizing logic circuits in digital systems. Sheffer strokes are often implemented as NAND gates in digital circuits. These gates are commonly used in electronic devices because they are simpler to implement and more cost-effective than other types of logic gates.

In this paper, we introduce and study the notion of fuzzy subalgebras (fuzzy ideals) with thresholds of Sheffer stroke Hilbert algebras is introduced, some properties of them are discussed, and its generalizations are proved. Further, we discuss the relations between fuzzy subalgebras (fuzzy ideals) with thresholds and their level subsets.

2 Preliminaries

In this section, basic definitions and notions about Sheffer stroke Hilbert. structures.

Definition 1. *(Sheffer, 1913) Let $H = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$$(S1)\ (x|(y|y))|(x|(y|y))) = y|x$$
$$(S2)\ (x|x)|((x|(y|y))|(x|(y|y)))) = x$$
$$(S3)\ x|((y|z)|(y|z)) = (((x|(y|y))|(x|(y|y))))|((x|(y|y))|(x|(y|y)))))|z$$
$$(S4)\ (x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.$$

Definition 2. *(Oner et al., 2021b) A Sheffer stroke Hilbert algebra is a structure $\langle H, | \rangle$ of type (2), in which H is a non-empty set and $|$ is a Sheffer stroke operation on H such that the following identities are satisfied for all $x, y, z \in H$:*

1. $(x|((P)|(P)))|(((Q)|((R)|(R)))|((Q)|((R)|(R)))) = x|(x|x)$, *where* $P := y|(z|z), Q := x|(y|y), R := x|(z|z)$,

2. *If* $x|(y|y) = y|(x|x) = x|(x|x)$ *then* $x = y$.

Proposition 1. *(Oner et al., 2021b) Let $\langle A, |\rangle$ be a Sheffer stroke Hilbert algebra. Then the binary relation $x \leq y$ if and only if $(x|(y|y)) = 1$ is a partial order on A.*

Definition 3. *(Oner et al., 2021b) A nonempty subset G of a Sheffer stroke Hilbert algebra X is called a subalgebra of X if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.*

Definition 4. *(Rosenfeld, 1971) Let f be a function from X to Y. If μ is a fuzzy set in X, then the fuzzy set β in Y is defined by*

$$\beta(y) = \begin{cases} \sup\limits_{t \in f^{-1}(y)} \{\mu(t)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is said to be the image of μ under f. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the preimage of β under f.

Definition 5. *(Rosenfeld, 1971) A fuzzy set μ in X is said to have the sup property if for any nonempty subset T of X, there exists $t_0 \in T$ such that $\mu(t_0) = \sup\limits_{t \in T} \mu(t)$.*

3 Fuzzy subalgebra/ideal of Sheffer stroke Hilbert algebra with thresholds

In this section, the study introduces the concepts of a fuzzy subalgebra and a fuuzy ideal of Sheffer stroke Hilbert algebras with thresholds. It's worth noting that unless explicitly stated otherwise, X refers to a Sheffer stroke Hilbert algebra.

Definition 6. *A fuzzy set μ in X is called a fuzzy subalgebra with thresholds ε and δ of X, where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, if $\forall x, y \in X$*

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} \geq \min\{\mu(x), \mu(y), \delta\}.$$

Proposition 2. *If μ is a fuzzy subalgebra with thresholds ε and δ of X, then $\max\{\mu(0), \varepsilon\} \geq \min\{\mu(x), \delta\}$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} \max\{\mu(0), \varepsilon\} &= \max\{\mu(1|1), \varepsilon\} \\ &= \max\{\mu((x|(x|x))|(x|(x|x))), \varepsilon\} \\ &\geq \min\{\mu(x), \mu(x), \delta\} \\ &= \min\{\mu(x), \delta\}. \end{aligned}$$

□

Definition 7. *A fuzzy set μ in X is called a fuzzy ideal with thresholds ε and δ of X, where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, if the following conditions are hold:*

$$(\forall x \in X)\left(\ \max\{\mu(0), \varepsilon\} \geq \min\{\mu(x), \delta\}\ \right), \tag{1}$$

$$(\forall\ x, y \in X)\left(\ \max\{\mu(x), \varepsilon\} \geq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\}\ \right). \tag{2}$$

Lemma 1. *If μ is fuzzy ideal with thresholds ε and δ of X, where $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$, then we have the following*

$$(\forall\ x, y \in H)\left(\ \ x \leq y\ \Rightarrow \min\{\mu(x), \delta\} \geq \max\{\mu(y), \varepsilon\}\ \right). \tag{3}$$

Proof. Let $x, y \in H$ be such that $x \leq y$. Then $(x|(y|y))|(x|(y|y)) = 0$ and so we obtain

$$\begin{aligned} \max\{\mu(y), \varepsilon\} &= \max\{\mu(0), \mu(y), \varepsilon\} \\ &= \max\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \varepsilon\} \\ &\leq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\} \\ &\leq \min\{\mu(x), \delta\}. \end{aligned}$$

□

Theorem 1. *Every fuzzy ideal with thresholds ε and δ of X is a fuzzy subalgebra with thresholds ε and δ.*

Proof. Assume that μ is a fuzzy ideal with thresholds ε and δ of X. Then $\max\{\mu(0), \varepsilon\} \geq \min\{\mu(x), \delta\}$ for all $x \in X$. Let $x, y \in X$. Then

Case (i): If $\mu(0) \leq \varepsilon$, then $\mu(x) \leq \varepsilon$ for all $x \in X$. Thus

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} = \varepsilon \geq \mu(x) \geq \min\{\mu(x), \mu(y), \varepsilon\}.$$

Case (ii): If $\mu(0) > \varepsilon$, then

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} \geq \min\{\mu(y), \delta\} = \min\{\mu(0), \mu(y), \delta\}.$$

If $\min\{\mu(0), \mu(y), \delta\} = \mu(y)$ or δ, then

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} \geq \min\{\mu(x), \mu(y), \delta\}.$$

If $\min\{\mu(0), \mu(y), \delta\} = \mu(0)$, then

$$\min\{\mu(0), \mu(y), \delta\} = \mu(0) = \max\{\mu(0), \varepsilon\} \geq \min\{\mu(y), \delta\} \geq \min\{\mu(x), \mu(y), \delta\}.$$

Hence μ is a fuzzy subalgebra with thresholds ε and δ of X. □

Theorem 2. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set μ in X is such that $\mu(x) \leq \varepsilon$ for all $x \in X$, then it is a fuzzy ideal (resp. fuzzy subalgebra) with thresholds ε and δ of X.*

Proof. Let $x \in X$. Then $\max\{\mu(0), \varepsilon\} = \varepsilon \geq \mu(x) = \min\{\mu(x), \delta\}$. Let $x, y \in X$. Then

$$\max\{\mu(x), \varepsilon\} = \varepsilon \geq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y)\} = \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\}.$$

Thus, μ is a fuzzy ideal with thresholds ε and δ of X.

Let $x, y \in X$. Then

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} = \varepsilon \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y), \delta\}.$$

Thus, μ is a fuzzy subalgebra with thresholds ε and δ of X. □

Theorem 3. *Let $\varepsilon, \delta \in [0, 1]$ with $\varepsilon < \delta$. If a fuzzy set μ in X is such that $\mu(x) \geq \delta$ for all $x \in X$, then it is a fuzzy ideal (resp. fuzzy subalgebra) with thresholds ε and δ of X.*

Proof. Let $x \in X$. Then

$$\max\{\mu(0), \varepsilon\} = \mu(0) \geq \delta = \min\{\mu(x), \delta\}.$$

Let $x, y \in X$. Then

$$\max\{\mu(x), \varepsilon\} = \mu(x) \geq \delta = \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\}.$$

Thus, μ is a fuzzy ideal with thresholds ε and δ of X.

Let $x, y \in X$. Then

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} = \mu((x|(y|y))|(x|(y|y))) \geq \delta = \min\{\mu(x), \mu(y), \delta\}.$$

Thus, μ is a fuzzy subalgebra with thresholds ε and δ of X. □

Theorem 4. *Let μ be a fuzzy set in X and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then μ is a fuzzy subalgebra with thresholds ε and δ of X if and only if for all $t \in (\varepsilon, \delta]$, the subset $U(\mu,t) = \{x \in X : \mu(x) \geq t\}$ is a subalgebra of X, if $U(\mu,t)$ is nonempty.*

Proof. Assume that μ is a fuzzy subalgebra with thresholds ε and δ of X. Let $t \in (\varepsilon, \delta]$ be such that $U(\mu,t) \neq \emptyset$ and let $x, y \in U(\mu,t)$. Then $\mu(x) \geq t$, $\mu(y) \geq t$, and $\delta \geq t$. Thus t is a lower bound of $\{\mu(x), \mu(y), \delta\}$. Since μ is a fuzzy subalgebra with thresholds ε and δ of X, we have

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} \geq \min\{\mu(x), \mu(y), \delta\} \geq t > \varepsilon.$$

Then, we obtain

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} = \mu((x|(y|y))|(x|(y|y))).$$

Since $\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} \geq t$, we get $\mu((x|(y|y))|(x|(y|y))) \geq t$. Hence

$$(x|(y|y))|(x|(y|y)) \in U(\mu,t).$$

Therefore, $U(\mu,t)$ is a subalgebra of X. Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(\mu,t)$ is a subalgebra of X, if $U(\mu,t)$ is nonempty. Let $x, y \in X$. Then $\mu(x), \mu(y) \in [0,1]$. Choose $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus $x, y \in U(\mu,t) \neq \emptyset$. By assumption, we have $U(\mu,t)$ is a subalgebra of A. So $(x|(y|y))|(x|(y|y)) \in U(\mu,t)$, that is, $\mu((x|(y|y))|(x|(y|y))) \geq t = \min\{\mu(x), \mu(y)\}$. So

$$\max\{\mu((x|(y|y))|(x|(y|y))), \varepsilon\} \geq \mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y)\} \geq \min\{\mu(x), \mu(y), \delta\}.$$

Hence, μ is a subalgebra with thresholds ε and δ of X. □

Theorem 5. *Let μ be a fuzzy set in X and $\varepsilon, \delta \in [0,1]$ with $\varepsilon < \delta$. Then μ is a fuzzy ideal with thresholds ε and δ of X if and only if for all $t \in (\varepsilon, \delta]$, $U(\mu,t)$ is an ideal of X, if $U(\mu,t)$ is nonempty.*

Proof. Assume that μ is a fuzzy ideal with thresholds ε and δ of X. Let $t \in (\varepsilon, \delta]$ be such that $U(\mu,t) \neq \emptyset$ and let $x \in U(\mu,t)$. Then $\mu(x) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{\mu(x), \delta\}$. Since μ is an fuzzy ideal with thresholds ε and δ of X, $\max\{\mu(x), \varepsilon\} \geq \min\{\mu(x), \delta\} \geq t > \varepsilon$. So $\max\{\mu(0), \varepsilon\} = \mu(0)$. Since $\max\{\mu(0), \varepsilon\} \geq t$, $\mu(0) \geq t$. Hence $0 \in U(\mu,t)$. Now, let $x, y \in X$ be such that $(x|(y|y))|(x|(y|y)), y \in U(\mu,t)$. Then $\mu((x|(y|y))|(x|(y|y))) \geq t$, $\mu(y) \geq t$ and $\delta \geq t$. Thus t is a lower bound of $\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\}$. Since μ is an fuzzy ideal with thresholds ε and δ of X,

$$\max\{\mu(x), \varepsilon\} \geq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\} \geq t \geq \varepsilon.$$

So $\max\{\mu(x), \varepsilon\} = \mu(x)$. Since $\max\{\mu(x), \varepsilon\} \geq t$, $\mu(x) \geq t$. Hence $x \in U(\mu,t)$. Hence $U(\mu,t)$ is an ideal of X. Conversely, assume that for all $t \in (\varepsilon, \delta]$, $U(\mu,t)$ is an ideal of X, if $U(\mu,t)$ is nonempty. Let $x \in X$. Then $\mu(x) \in [0,1]$. Choose $t = \mu(x)$. Then $\mu(x) \geq t$. Thus $x \in U(\mu,t) \neq \emptyset$. By assumption, we have $U(\mu,t)$ is an ideal of X. So $0 \in U(\mu,t)$, that is, $\mu(0) \geq t = \mu(x)$. Hence $\max\{\mu(0), \varepsilon\} \geq \mu(0) \geq \mu(x) \geq \min\{\mu(x), \delta\}$. Now, let $x, y \in X$. Then $\mu((x|(y|y))|(x|(y|y))), \mu(y) \in [0,1]$. Choose $t = \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y)\}$. Then $\mu((x|(y|y))|(x|(y|y))) \geq t$ and $\mu(y) \geq t$. Thus $(x|(y|y))|(x|(y|y)), y \in U(\mu,t) \neq \emptyset$. By assumption, we have $U(\mu,t)$ is an ideal of X. So $x \in U(\mu,t)$, that is, $\mu(x) \geq t = \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y)\}$. Hence

$$\max\{\mu(x), \varepsilon\} \geq \mu(x) \geq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y)\} \geq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y), \delta\}.$$

Therefore, μ is an ideal with thresholds ε and δ of X. □

Lemma 2. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke Hilbert algebras and let $f : A \to B$ be a surjective homomorphism. Let μ be an f-invariant fuzzy set in A with sup property. For any $a, b \in B$, there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0)$, $\beta(b) = \mu(b_0)$, and $\beta(a|_B b) = \mu(a_0|_A b_0)$.*

Theorem 6. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Hilbert algebras and let $f : A \to B$ be a surjective homomorphism. Then the following statements hold:*

1. *if μ is an f-invariant fuzzy subalgebra with thresholds ε and δ of A with sup property, then β is a fuzzy subalgebra with thresholds ε and δ of B,*

2. *if μ is an f-invariant fuzzy ideal with thresholds ε and δ of A with sup property, then β is a fuzzy ideal with thresholds ε and δ of B.*

Proof. (1). Assume that β is an f-invariant fuzzy subalgebra with thresholds ε and δ of A with sup property. Let $a, b \in B$. Then there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\beta(a) = \mu(a_0)$, $\beta(b) = \mu(b_0)$, and

$$\beta((a|_B(b|_Bb))|_B(a|_B(b|_Bb))) = \mu((a_0|_A(b_0|_Ab_0))|_A(a_0|_A(b_0|_Ab_0))).$$

Thus $\max\{\beta((a|_B(b|_Bb))|_B(a|_B(b|_Bb))), \varepsilon\} = \max\{\mu((a_0|_A(b_0|_Ab_0))|_A(a_0|_A(b_0|_A b_0))), \varepsilon\} \geq \min\{\mu(a_0), \mu(b_0), \delta\} = \min\{\beta(a), \beta(b), \delta\}$. So, β is a fuzzy subalgebra with thresholds ε and δ of B.
(2). Assume that β is an f-invariant fuzzy ideal with thresholdsε and δ of A with sup property. Since $f(0_A) = 0_B$, we have $f^{-1}(0_B) \neq \emptyset$. Then there exists $x_1 \in f^{-1}(0_B)$ such that $\beta(x_1) = \mu(0_B)$. Thus $f(x_1) = 0_B = f(0_A)$. Since β is f-invariant, $\beta(x_1) = \mu(0_A)$. So, $\beta(0_A) = \mu(0_B)$. Let $y \in B$. Since f is surjective, we have $f^{-1}(y) \neq \emptyset$. Then there exists $x \in f^{-1}(y)$ such that $\mu(x) = \beta(y)$. Thus $\max\{\beta(0_B), \varepsilon\} = \max\{\mu(0_A), \varepsilon\} \geq \min\{\mu(x), \delta\} = \min\{\beta(y), \delta\}$. Now let $a, b \in B$. Then there exist $a_0 \in f^{-1}(a)$ and $b_0 \in f^{-1}(b)$ such that $\mu(a_0) = \beta(a)$, $\mu(b_0) = \beta(b)$ and $\mu((a_0|(b_0|b_0))|(a_0|(b_0|b_0))) = \beta((a|(b|b))|(a|(b|b)))$. Thus

$$\max\{\beta(a), \varepsilon\} = \max\{\mu(a_0), \varepsilon\} \geq \min\{\mu((a_0|(b_0|b_0))|(a_0|(b_0|b_0))), \mu(b_0), \delta\} =$$
$$= \min\{\beta((a|(b|b))|(a|(b|b))), \beta(b), \delta\}.$$

Therefore, β is a fuzzy ideal with thresholds ε and δ of B. □

Theorem 7. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Hilbert algebras and let $f : A \to B$ be a homomorphism. Then the following statements hold:*

1. *if β is a fuzzy subalgebra with thresholds ε and δ of B, then μ is a fuzzy subalgebra with thresholds ε and δ of A,*

2. *if β is a fuzzy ideal with thresholds ε and δ of B, then μ is a fuzzy ideal with thresholds ε and δ of A.*

Proof. (1). Assume that β is a fuzzy subalgebra with thresholds ε and δ of B. Let $x, y \in A$. Then

$$\begin{aligned}
&\max\{\mu((x|_A(y|_Ay))|_A(x|_A(y|_Ay))), \varepsilon\} \\
&\quad = \max\{(\beta \circ f)((x|_A(y|_Ay))|_A(x|_A(y|_Ay))), \varepsilon\} \\
&\quad = \max\{(\beta(f((x|_A(y|_Ay))|_A(x|_A(y|_Ay))), \varepsilon\} \\
&\quad = \max\{(\beta(f(x)|_B(f(y)|_Bf(y)))|_B(f(x)|_B(f(y)|_Bf(y)))), \varepsilon\} \\
&\quad \geq \min\{\beta(f(x)), \beta(f(y)), \delta\} \\
&\quad = \min\{(\beta \circ f)(x), (\beta \circ f)(y), \delta\} \\
&\quad = \min\{\mu(x), \mu(y), \delta\}.
\end{aligned}$$

Thus, μ is a fuzzy subalgebra with thresholds ε and δ of A.
(2). Assume that β is a fuzzy ideal with thresholds ε and δ of B. Let $x \in A$. Then

$$\begin{aligned}\max\{\mu(0_A), \varepsilon\} &= \max\{(\beta \circ f)(0_A), \varepsilon\} \\ &= \max\{(\beta(f(0_A)), \varepsilon\} \\ &= \max\{\beta(0_B), \varepsilon\} \\ &\geq \min\{\beta(f(x)), \delta\} \\ &= \min\{(\beta \circ f)(x), \delta\} \\ &= \min\{\mu(x), \delta\}.\end{aligned}$$

Let $x, y \in A$. Then we have

$$\begin{aligned}\max\{\mu(x), \varepsilon\} &= \max\{(\beta \circ f)(x), \varepsilon\} \\ &= \max\{\beta(f(x)), \varepsilon\} \\ &\geq \min\{\beta((f(x)|_B(f(y)|_B f(y)))|_B(f(x)|_B(f(y)|_B f(y)))), \beta(f(y)), \delta\} \\ &= \min\{(\beta \circ f)((x|_A(y|_A y))|_A(x|_A(y|_A y))), (\beta \circ f)(y), \delta\} \\ &= \min\{\mu((x|_A(y|_A y))|_A(x|_A(y|_A y))), \mu(y), \delta\}.\end{aligned}$$

Thus, μ is a fuzzy ideal with thresholds ε and δ of A. □

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